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Eigenvalue Error Analysis of Viscously Damped Structures Using a Ritz Reduction Method

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The efficient solution of the eigenvalue problem that results from inserting passive dampers with variable stiffness and damping coefficients into a structure is addressed. Eigenanalysis of reduced models obtained by retaining a number of normal modes augmented with Ritz vectors corresponding to the static solutions resulting from the load patterns introduced by the dampers has been empirically shown to yield excellent approximations to the full eigenvalue problem. An analysis of this technique in the case of a single damper is presented. A priori and a posteriori error estimates are generated and tested on numerical examples. Comparison theorems with modally truncated models and a Markov parameter matching reduced-order model are derived. These theorems corroborate the heuristic that residual flexibility methods improve low-frequency approximation of the system. The analysis leads to other techniques for eigenvalue approximation. Approximate closed-form solutions are derived that include a refinement to eigenvalue derivative methods for approximation. An efficient Newton scheme is also developed. A numerical example is presented demonstrating the effectiveness of each of these methods.

Introduction

A PROBLEM of considerable importance in the development of technology for future lightly damped large space structures is the analysis and optimization of damping introduced by passive elements placed in these structures.[1] Passive damping introduced by these devices is an effective mechanism for reducing peak responses in the vicinity of the resonant frequencies for lightly damped systems.[2] This not only enhances the stability of the open-loop system, but facilitates the implementation of aggressive active control strategies to achieve greater performance. This strategy is currently being pursued on a series of test beds at the Jet Propulsion Laboratory (JPL) to demonstrate nanometer level optical pathlength control on a flexible structure that emulates a proposed space interferometer.[3]

The effectiveness of viscous elements in introducing damping is a function of their number, their placement in the structure, and their physical parameters. This paper was motivated by the considerations involved in determining these variables; more specifically, with overcoming the large numerical problems encountered when computing the damping in specific modes that are targeted in the control design process.

Given the variables that characterize a damper's performance (location, stiffness, and damping coefficient), the brute force approach to this computation is extremely expensive since it involves solving a large unsymmetric eigenvalue problem (the midsize JPL test-bed model contains approximately 250 degrees of freedom). In optimization applications where repeated solutions of these systems are required, it is necessary to substantially reduce the full-order model. Reduced-order models obtained by modal truncation were discovered to be quite inaccurate in their damping prediction for the test-bed application. Order of magnitude discrepancies occurred in the prediction of damping in some modes between the full and modally reduced models.

Subsequently, it was discovered that the addition of a single vector consisting of a linear combination of the truncated modes to statically correct the modally reduced model produced very high fidelity frequency and damping estimates over a wide range of damper parameters. [This method will be referred to in the paper as the Ritz reduction method or simply the Ritz method. And the augmenting vector(s) will be referred to as the Ritz vector(s).] This result is not altogether surprising, as the use of static correction modes to improve model fidelity has been a standard approach in component mode synthesis techniques for many years.[4-6] Later, their use appeared for the purpose of accelerating convergence in the transient analysis of systems under various loading conditions.[7,8] More recently, several articles have investigated the use of static correction modes in applications to systems similar to the one considered here.[9-12] In Ref. 9, the Ritz reduction method is used to enhance eigenvalue and eigenvector convergence for systems subject to external or interactive forces. Dramatic proof of the effectiveness of the method is illustrated on an example of a pinned-pinned beam with torsional spring. Sandridge and Haftka[10] apply these methods to velocity feedback control systems and are able to significantly reduce the sensitivity of the closed-loop eigenvalues. The inclusion of Ritz vectors has an analog in the systems theory model reduction literature as a moment matching method.[13] This approach was taken in Ref. 11 to produce a moment matching method for systems in second-order form. The resulting reduced model has the advantage that it is still in second-order form and, thus, preserves the symmetry and stability properties of the second-order formulation.[11]

Although considerable empirical evidence has accumulated to support the use of static correction modes, little analysis exists to quantify or draw specific comparisons regarding their benefits. Most of the analysis in this area centers around the motivation for using the modes to accelerate convergence. Insightful analysis in this direction is due to Baruh and Tadikonda,[14] who discuss several related problems of slow convergence in the context of a Gibb's phenomenon.

In this paper, specific analysis is presented to justify, and to an extent, quantify the effectiveness of static correction modes. Beginning with the perspective that the inclusion of static modes is equivalent to matching the zeroth moment between the full and reduced models, an analytical formula is developed to study eigenvalue error estimates for the reduced plant in the case where a single damper is added to the nominal

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system. This analysis leads to a priori error estimates for the Ritz reduction method for systems undergoing small perturbations. When coupled with a similar a priori error bound for modally reduced models, the ratio for the two error estimates gives an indication of the accuracy improvement attained by the method. The same analytical expression also leads to an a posteriori eigenvalue error estimate for the Ritz reduction method. Because these estimates are easily obtained numerically, they can be quite useful in a number of applications. Several numerical examples indicate these estimates to be quite tight. A more global comparison between modally truncated and statically corrected models that is independent of the size of the stiffness perturbations and valid for lightly damped systems is presented. In the same spirit, a theorem is developed for comparing reduced models obtained by augmenting vectors to match the zeroth system moment (the Ritz method) vs matching the system's first Markov parameter. (In the undamped case, the Ritz reduced model also matches the first system moment.) It is shown in the undamped case, and extended to the lightly damped case, that independent of the stiffness change to the nominal system, the Ritz reduction method leads to a better approximation of all eigenvalues of the system. These theorems substantiate the heuristic that static correction, by virtue of its moment matching property, improves the low-frequency approximation of the system.

Because the original motivation for studying the Ritz reduction method arose from considerations stemming from the optimization problem, related analysis is presented for other approaches to eigenvalue approximation that are suited for optimization. Here, the use of eigenvalue derivatives as proposed in Ref. 15 and other alternatives are briefly discussed.

A number of numerical examples involving the JPL test-bed structure are presented to support the analysis given in the paper. In addition, the numerical studies include a multiple damper case to demonstrate the effectiveness of the Ritz reduction technique in this setting as well.

Modeling of Viscously Damped Structures Using the Ritz Reduction Method

Throughout this study, it is assumed that the dynamics of the undamped structure can be described by a linear second-order matrix differential equation of the form:

$$M\ddot{z} + Kz = B_d d \tag{1}$$

Here, z denotes the n-dimensional vector of generalized coordinates; d the p-dimensional external forcing input vector; M the $n \times n$ symmetric, positive definite mass matrix; K the $n \times n$ symmetric, positive definite stiffness matrix; and B_d the $n \times p$ forcing input influence matrix.

Assume that a discrete passive damper is placed between two nodal points in the structure replacing the original structural element. The passive damper is modeled as a device that applies a force at the nodal points with equal magnitude but in opposite directions and proportional to the relative displacement and velocity between the nodal points. The dynamic structural model incorporating the damper actuator force u is written as,

$$M\ddot{z} + Kz = \tilde{b}u + B_d d \tag{2}$$

where the vector $\tilde{b}$ represents the influence vector associated with u. The force u generated by the damper is modeled as a constant linear combination of collocated position and velocity feedback so that

$$u = -(k_p y_p + k_v y_v) \tag{3}$$

with

$$y_p = \tilde{b}^T z \tag{4}$$

$$y_v = \tilde{b}^T \dot{z} \tag{5}$$

where y_p and y_v denote the position and velocity measurements, respectively, and k_v denotes the damping rate, which is always taken as a nonnegative quantity to ensure stability. The parameter k_p is only required to be greater than or equal to the value $-k_e$, where k_e denotes the stiffness of the structural element that has been replaced by the damper. When $-k_e \le k_p < 0$, the structure is softened, whereas $k_p > 0$ causes the structure to be stiffened.

Combining Eqs. (2–5), the dynamic structural model with the inclusion of the single damper can be represented as

$$M\ddot{z} + k_v\tilde{b}\tilde{b}^T\dot{z} + (K + k_p\tilde{b}\tilde{b}^T)z = B_d d \tag{6}$$

The model reduction method considered in this paper is a second-order reduction technique based on reducing the number of generalized coordinates of the system via a transformation of the form $z = \mathfrak{Z}q \in R^m$ with $m < n$. Applying the transformation $\mathfrak{Z}$ to Eq. (6) results in the reduced-order model

$$\mathfrak{Z}^T M\mathfrak{Z}\ddot{q} + k_v\mathfrak{Z}^T\tilde{b}\tilde{b}^T\mathfrak{Z}\dot{q} + \mathfrak{Z}^T(K + k_p\tilde{b}\tilde{b}^T)\mathfrak{Z}q = \mathfrak{Z}^T B_d d \tag{7}$$

The corresponding open-loop system associated with Eq. (7) is

$$\mathfrak{Z}^T M\mathfrak{Z}\ddot{q} + \mathfrak{Z}^T K\mathfrak{Z}q = \mathfrak{Z}^T\tilde{b}u + \mathfrak{Z}^T B_d d \tag{8}$$

with measurement

$$\begin{bmatrix} y_p \\ y_v \end{bmatrix} = \begin{bmatrix} \tilde{b}^T\mathfrak{Z}q \\ \tilde{b}^T\mathfrak{Z}\dot{q} \end{bmatrix} \tag{9}$$

The system (7) is recovered from the feedback control law $u = -(k_p y_p + k_v y_v)$, just as in Eq. (3). The transformation $\mathfrak{Z}$ in Eq. (7) is chosen so that the open-loop transfer function from u to y in the full and reduced systems, Eqs. (2–5) and (8) and (9), respectively, agree at zero frequency. An equivalent interpretation of this requirement is that the reduced model is statically correct in the sense that the static deformation due to the force exerted by the damper is captured by the model.

This property alone does not uniquely define $\mathfrak{Z}$. The specific transformation that will be analyzed in this paper is constructed in the following way: Let Φ_r denote the matrix whose columns are comprised for the first r eigenvectors of the undamped system, and assuming no rigid-body modes, let $\psi = K^{-1}\tilde{b}$. Then $\mathfrak{Z}$ is defined as $\mathfrak{Z} = [\Phi_r\ \psi]$. The ψ will be referred to as the Ritz vector (or, equivalently, the forced mode). We note that Ritz vectors can be developed for systems with rigid-body modes as well.[9] It is not difficult to show (see, e.g., Ref. 7) that $\mathfrak{Z}$ has the desired static correction property.

The fundamental question is how well does Eq. (7) approximate the full system (6)? In the context of optimizing damper locations and parameter values, this question is more specifically, how accurately do the poles of Eq. (7) approximate the poles of Eq. (8)? This question is taken up in the following section.

Analysis

Let $\Phi \in R^{n\times n}$ denote the modal matrix so that

$$\Phi^T M\Phi = I, \qquad \Phi^T K\Phi = D$$

with $D = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $0 < \omega_1 < \ldots < \omega_n$. Thus, it is assumed that the system does not contain rigid-body modes and has no repeated frequencies. Applying the transformation Φ to the system (6) results in

$$\ddot{q} + k_v bb^T\dot{q} + (D + k_p bb^T)q = g \tag{10}$$

where $q = \Phi z$, k_p and k_v are real numbers with $k_v \ge 0$, $b = \Phi^T\tilde{b}$, and $g = \Phi^T B_d d$ is a forcing function. The key to the analysis is that the system (10) with a single damper element can be analyzed from the perspective of a rank-one perturbation to the matrix D.

The following notations and constants will be used throughout this section. The standard orthonormal basis in R^n will be denoted by $\{e_i\}_{i=1}^n$, so that whenever a vector $b \in R^n$ is written as a column vector, $b = [b_1, \ldots, b_n]^T$, then $b_i = \langle b, e_i\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on R^n. The constants γ_i, μ_1, and μ_2 are defined as,

$$\gamma_i = \frac{b_i^2}{\sum_{j=r+1}^n b_j^2}, \qquad i = r+1, \ldots n \tag{11}$$

$$\mu_1 = \sum_{j=r+1}^n \frac{\gamma_j}{\omega_j^2} \tag{12}$$

$$\mu_2 = \sum_{j=r+1}^n \frac{\gamma_j}{\omega_j^4} \tag{13}$$

Next, define the set of orthonormal vectors $\{f_i\}_{i=1}^{r+1}$ by $f_i = e_i$, $i = 1, \ldots, r$, and

$$f_{r+1} = \frac{\sum_{j=r+1}^n \langle D^{-1}b, e_j\rangle e_j}{\left[\sum_{j=r+1}^n \langle D^{-1} b, e_j\rangle^2\right]^{1/2}}$$

$$= \frac{\sum_{j=r+1}^n (b_j/\omega_j^2) e_j}{\left[\sum_{j=r+1}^n (b_j/\omega_j^2)^2\right]^{1/2}}$$

The vector f_{r+1} is, thus, the Ritz vector ψ projected onto the truncated modes. The reduced-order model with the augmenting term is then defined as,

$$I\ddot{q} + U^T[k_v bb^T\dot{q} + (D + k_p bb^T q)]U = U^T g \tag{14}$$

where U denotes the $n \times (r+1)$ matrix with columns consisting of the vectors f_i, $i = 1, \ldots, r+1$, and I denotes the $r+1 \times r+1$ identity matrix.

The following theorem, which is the second-order analog of a result in Golub and Van Loan,[16] is fundamental to the analysis of the succeeding sections.

Theorem 1. Consider the nonlinear eigenvalue problem

$$\det M(\lambda) = 0 \tag{15}$$

where

$$M(\lambda) = \lambda^2 I + D + (\lambda k_v + k_p) bb^T$$

with $b_i \neq 0$ for all i, and k_p, $k_v \in R$ with $k_p k_v \neq 0$. If $D = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ with $0 < \omega_1 < \cdots < \omega_n$, then $\pm i\omega_j$ is not a solution of $\det M(\lambda) = 0$ for $j = 1, \ldots, n$. Furthermore, the solutions of Eq. (15) are the zeros of the function $d(\lambda)$,

$$d(\lambda) = 1 + (\lambda k_v + k_p) \sum_{j=1}^n \frac{b_j^2}{\lambda^2 + \omega_j^2}$$

Since the reduced-order system (14) can be viewed as a rank-one perturbation of $U^T DU$, an analogous rational function d_u whose zeros are the poles of Eq. (14) can be developed for it. In the following theorem, the error $d - d_u$ is quantified. This difference will then be used to derive a priori and a posteriori error estimates for eigenvalue approximations obtained by the Ritz reduction method.

Theorem 2. The poles of the system (14) are the zeros of the function

$$d_u(\lambda) = 1 + (\lambda k_v + k_p)\left\{\sum_{j=1}^r \frac{b_j^2}{\lambda^2 + \omega_j^2} + \frac{\langle b, f_{r+1}\rangle^2}{\langle Df_{r+1}, f_{r+1}\rangle + \lambda^2}\right\}$$

And for $|\lambda|^2 < \omega_{r+1}^2$, $d - d_u$ is analytic with

$$d(\lambda) - d_u(\lambda) = -\lambda^4(\lambda k_v + k_p) \sum_{j=r+1}^n b_j^2 \left\{\frac{\mu_2^2/\mu_1}{1 + \lambda^2\mu_2/\mu_1} - \sum_{k=r+1}^n \frac{\gamma_k/\omega_k^6}{1 + \lambda^2/\omega_k^2}\right\} \tag{16}$$

Consider now any vector $\tilde{f}_{r+1}$ augmenting the first r modes, $e_1, \ldots, e_r$, say,

$$\tilde{f}_{r+1} = \sum_{j=r+1}^n a_j e_j$$

with $\Sigma a_j^2 = 1$ such that the associated scalar function $\tilde{d}$ (cf. Theorem 1) approximates d to fourth order, i.e., $d - \tilde{d}$ has a zero of order 4 about zero. Theorem 1 shows that

$$\tilde{d} = 1 + (\lambda k_v + k_p)\left\{\sum_{j=1}^r \frac{b_j^2}{\omega_j^2 + \lambda^2} + \frac{\langle b, \tilde{f}_{r+1}\rangle^2}{\langle D\tilde{f}_{r+1}, \tilde{f}_{r+1}\rangle + \lambda^2}\right\}$$

Then since $\tilde{d}$ approximates d to fourth order, the identities

$$\langle b, f_{r+1}\rangle^2 = \langle b, \tilde{f}_{r+1}\rangle^2$$

$$\langle Df_{r+1}, f_{r+1}\rangle = \langle D\tilde{f}_{r+1}, \tilde{f}_{r+1}\rangle$$

must hold. Hence, $\tilde{d} = d_u$, and the following uniqueness property of the Ritz reduced model is deduced: Any reduced-order model obtained by augmenting the first r eigenvectors with a single vector such that the associated rational function $\tilde{d}$ has the property that $d - \tilde{d}$ has a zero of order 4 at zero yields the same approximate eigenvalues as the reduced system (14) irrespective of the values of the parameters k_p and k_v.

Now that an analytical expression for $d - d_u$ has been established, this information will next be used to estimate how closely the poles of Eq. (14) approximate the poles of Eq. (10) in the case that k_v and k_p lead to small perturbations of the nominal system. There are few general tools available to carry out this analysis. In the following, the Newton-Kantorovich theorem[17] is used in conjunction with Theorem 2 to obtain these estimates.

For $|k_v|$ and $|k_p|$ sufficiently small, let $\lambda(k_v, k_p)$ and $\lambda_u(k_v, k_p)$ denote the perturbed kth eigenvalue of systems (10) and (14), respectively, corresponding to the parameters k_v and k_p. Then $\lambda(\cdot)$ and $\lambda_u(\cdot)$ are continuous functions of k_v and k_p in a neighborhood of $k_v = k_p = 0$ with $\lambda(0, 0) = \lambda_u(0, 0) = i\omega_k$. This observation is used in the following theorem.

Theorem 3. Let λ and λ_u be defined as aforementioned. For $|k_v|$ and $|k_p|$ sufficiently small,

$$|\lambda - \lambda_u| \leq |\lambda_u - i\omega_k| \frac{\omega_k^3(\omega_k k_v + |k_p|)}{\omega_{r+1}^4(\omega_{r+1}^2 - \omega_k^2)} \sum_{i=r+1}^n b_i^2 \tag{17}$$

Deleting the Ritz vector $\tilde{f}_{r+1}$ from the reduced model (14) and applying the same argument in the theorem to the resulting modally reduced model leads to an a priori estimate of the form

$$|\lambda_{\mathrm{modal}} - \lambda| \leq |\lambda_{\mathrm{modal}} - i\omega_k| \frac{(\omega_k k_v + |k_p|)}{\omega_{r+1}^2 - \omega_k^2} \sum_{i=r+1}^n b_i^2 \tag{18}$$

Thus, in terms of a priori error analysis, the inclusion of the Ritz vector f_{r+1} results in an error estimate improvement of a factor of $\mathcal{O}(\omega_k^3/\omega_{r+1}^4)$ over the modally reduced model.

The estimate (17) gives considerable insight into the various factors that affect the approximation error (e.g., the ratio between the magnitudes of the estimated eigenvalue and the highest mode retained, the magnitude of the difference between the nominal and estimated eigenvalue, etc.). In addition to these, Eq. (17) can be quite useful as an a posteriori error estimate for the reduced-order model.

To see this, consider the function

$$f(\lambda) = (\lambda^2 + \omega_k^2)d(\lambda)$$

with $d(\lambda)$ defined as in Theorem 1. In the proof of this theorem (see the Appendix), it is shown that if

$$\frac{|f(\lambda_u)|}{|f'(\lambda_u)|^2}|f''(\lambda)| < \frac{1}{2} \tag{19}$$

for all λ in $N(\lambda_u,\rho)$ with

$$\rho > \frac{1}{h}(1 - \sqrt{1-2h})\frac{|f(\lambda_u)|}{f'(\lambda_u)}$$

where h is the supremum over $\lambda \epsilon N(\lambda_u, \rho)$ on the left side of Eq. (19), then

$$|\lambda_u - \lambda| < \rho_0 \tag{20}$$

$$\rho_0 = \frac{2|f(\lambda_u)|}{|f'(\lambda_u)|}$$

The quantity ρ_0 is readily computed, while an estimate of $\sup_{\lambda\epsilon N(\lambda_u,\rho_0)}|f''(\lambda)|$ can also be obtained. The accuracy of the a posteriori estimates obtained in this fashion for the Ritz reduction method will be investigated later.

The arguments in Theorem 3 indicate that significantly superior a priori eigenvalue error estimates can be obtained by augmenting the modally reduced model with the Ritz vector corresponding to the static solution resulting from the load introduced by the damper. However, the method of proof relies on the Newton-Kantorovich theorem, which in itself is a local result, and correspondingly, the result of Theorem 3 is also local (small stiffness and damping changes from nominal values). A more global comparison between these reduction methods is next investigated, initially under a zero damping assumption, which will subsequently be relaxed to a light damping assumption.

Consider now a reduced-order model obtained from retaining the first r eigenvectors and a single high-frequency mode e_s with $s > r+1$. With the assumption that $k_v = 0$, Theorem 1 implies that the zeros of the rational function

$$d_s(\lambda) = 1 + k_p\left[\frac{b_s^2}{\omega_s^2 + \lambda^2} + \sum_{j=1}^{r}\frac{b_j^2}{\omega_j^2 + \lambda^2}\right]$$

coincide with the poles of the reduced system

$$I\ddot{q} + V^T[D + k_p bb^T]Vq = V^T g \tag{21}$$

where V is the $n \times (r+1)$ matrix with columns $e_1, \ldots, e_r, e_s$. To simplify the notation in the statements and proofs of the results to follow, the following functions related to d, d_u, and d_s are introduced:

$$g(t) = 1 + k_p\sum_{j=1}^{n}\frac{b_j^2}{\omega_j^2 - t}$$

$$g_u(t) = 1 + k_p\left[\sum_{j=1}^{r}\frac{b_j^2}{\omega_j^2 - t} + \frac{\langle b, f_{r+1}\rangle^2}{\langle Df_{r+1}, f_{r+1}\rangle - t}\right] \tag{22}$$

$$g_s(t) = 1 + k_p\left[\frac{b_s^2}{\omega_s^2 - t} + \sum_{j=1}^{r}\frac{b_j^2}{\omega_j^2 - t}\right] \tag{23}$$

Note that, when $k_v = 0$, t^* is a zero of g if and only if $t^* = |\lambda^*|^2$ where λ^* is a zero of d. The analogous relations hold between the zeros of g_u and d_u and between g_s and d_s. In the following theorem, for each k_p, the zeros of g are denoted by $t^j(k_p)$ with $t^1 < t^2 \ldots < t^n$. The zeros of g_u are similarly denoted by $t_u^j(k_p)$, and the zeros of g_s by $t_s^j(k_p)$.

Theorem 4. There exists an index k, $1 \le k \le r+1$, independent of k_p such that for any index $j < k$,

$$|t^j(k_p) - t_u^j(k_p)| \le |t^j(k_p) - t_s^j(k_p)| \tag{24}$$

and for $j > k$,

$$|t^j(k_p) - t_s^j(k_p)| \le |t^j(k_p) - t_u^j(k_p)| \tag{25}$$

If $\gamma_s\mu_2/\mu_1 - \mu_1 > 0$, then $k = r+1$. If $\gamma_s\mu_2/\mu_1 - \mu_1 < 0$, then k is determined by the relation

$$\omega_k^2 < \frac{\mu_1\omega_s^2 - \gamma_s}{\mu_1 - \gamma_s\mu_2/\mu_1} < \omega_{k+1}^2 \tag{26}$$

The crux of this result is that when $k_v = 0$, there exists an index k such that the first $k-1$ poles of Eq. (10) are better approximated by the reduced system (14) than by Eq. (21). In a complementary fashion, the $k+1$st through the rth poles of Eq. (21) are better approximations to the corresponding poles of Eq. (10) than the approximations obtained from Eq. (14). This result holds independently of the value of k_p, and furthermore, the index k can be determined in an a priori fashion via Eq. (26).

As a corollary to this result, note that in the proof of the theorem strict inequalities hold in Eqs. (24) and (25) when $b_i \neq 0$ for all i. Thus, in this case, continuity of the system eigenvalues with respect to k_v implies that for sufficiently small $|k_v|$ these inequalities still hold. That is, the theorem holds for lightly damped systems as well.

Corollary 5. Suppose $b_i \neq 0$ for all i. Then there is a real valued function $r(k_p) > 0$ such that the relations in the theorem hold whenever $|k_v| < r(k_p)$ where the poles in Eqs. (24) and (25) are interpreted as the damped poles introduced by the nonzero term k_v.

The feedback representation (2-5) leads to an interpretation of the Ritz reduction (or forced mode) method as a moment matching method.[11] The addition of the vector $K^{-1}\tilde{b}$ to the modal reducing transformation causes the reduced-model transfer function between u and y to match the zero frequency component of the full-order transfer function. This is equivalent to matching the zeroth-order terms of the Taylor series expansions of the reduced- and full-order transfer functions. Heuristically, moment matching approximation methods lead to a good approximation of the low-frequency behavior of the system. At the opposite end of the spectrum are the model reduction methods based on matching the system Markov parameters.[13] This involves matching the terms in the Laurent series expansion of the system. In the context of second-order model reduction techniques, the first Markov parameter can be matched by augmenting the modally reduced model with the vector $\tilde{b}$. Thus, the reducing transformation in this case is $\mathfrak{I}_{\mathrm{MARKOV}} = [\Phi_r\ \tilde{b}]$. Heuristically, a reduced-order model obtained by matching Markov parameters yields a good approximation to the high-frequency behavior of the system. The theorem and corollary below show that, in the lightly damped case, augmenting the modally reduced model to capture the first moment by including the Ritz vector ψ leads to a better approximation of all poles than by augmenting the model to match the first Markov parameter obtained by including the vector $\tilde{b}$.

Let b^* denote the orthogonal projection of the vector b onto the subspace spanned by $\{e_{r+1}, \ldots, e_n\}$,

$$b^* = \frac{\sum_{j=r+1}^{n} b_j e_j}{\sum_{j=r+1}^{n} b_j^2}$$

and consider the reduced-order model

$$I\ddot{q} + U_b^T[k_v bb^T\dot{q} + (Dq + k_p bb^T q)]U_b = U_b^T g \tag{27}$$

where U_b is the $n \times (r+1)$ matrix with columns $e_1, \ldots, e_r, b^*$. For $k_v = 0$, let $t_b^j(k_p)$ denote the jth pole of Eq. (25).

Theorem 6. In the notation of the previous theorem,

$$|t^j(k_p) - t_u^j(k_p)| < |t^j(k_p) - t_{b*}^j(k_p)|$$

for all k_p and all indices j.

For lightly damped systems, a corollary analogous to Corollary 5 holds for these approximations.

Corollary 7. There exists a real valued function $r(k_p) > 0$ such that whenever $k_v < r(k_p)$

$$|\lambda^j(k_p, k_v) - \lambda_u^j(k_p, k_v)| < |\lambda^j(k_p, k_v) - \lambda_{b*}^j(k_p, k_v)|$$

where $\lambda_{b*}^j(k_p, k_v)$ denotes the perturbed jth pole of the system (25) (cf. Corollary 5).

Consider the collocated system

$$\ddot{q} + Dq = -bu \tag{28}$$

$$y_0 = k_p b^T q + k_v b^T \dot{q} \tag{29}$$

As the transfer function of this system is easily determined to be $d(s) - 1$, where $d(s)$ is the rational function defined in Theorem 1, the results of this section concerning approximating the poles of Eq. (10) have a dual interpretation in terms of approximating the zeros of the system (28) and (29). In a parallel development to the eigenvalue approximation problem, substantial empirical evidence indicates that augmenting modally reduced models with static correction modes for the aforementioned feedback system leads to significant improvement in approximate system zeros for these systems.[18] In the context of control system design, accurate knowledge of system zeros is extremely important since it is a major factor in determining achievable closed-loop performance and robustness.

Table 1 Frequency prediction (Hz) of the damped system ($k_{p133} = 5000.72$ lb/in., $k_{v133} = 400$ lb-s/in.)

		Damped		
Mode number	Undamped	249 modes (full model)	12 modes plus 1 Ritz vector	13 modes (truncation)
1	0.7427	0.7417	0.7417	0.7424
2	5.4263	5.2315	5.2315	5.2618
3	7.4565	7.1353	7.1353	7.2170
4	11.6777	11.5462	11.5462	11.4116
5	17.4248	17.4846	17.4846	17.2939
6	20.8423	20.9812	20.9811	20.4959
7	31.1387	31.1477	31.1477	31.1301
8	40.4695	40.6585	40.6585	40.3343
9	54.3509	54.6942	54.6950	54.1262
10	65.6963	65.9101	65.9119	65.5360
11	68.8744	68.8930	68.8932	68.8569
12	80.8372	80.8419	80.8420	80.8338

Table 2 Damping prediction (%) of the damped system ($k_{p133} = 5000.72$ lb/in., $k_{v133} = 400$ lb-s/in.)

	Damped		
Mode number	249 modes (full model)	12 modes plus 1 Ritz vector	13 modes (truncation)
1	0.0234	0.0234	0.0023
2	8.2316	8.2316	1.2536
3	6.8431	6.8431	1.4860
4	4.9212	4.9211	1.5728
5	1.4621	1.4622	0.8471
6	3.3098	3.3099	1.7420
7	0.0440	0.0440	0.0468
8	0.4373	0.4382	0.7900
9	0.4033	0.4068	1.2478
10	0.1710	0.1749	0.7056
11	0.0146	0.0150	0.0520
12	0.0023	0.0025	0.0157

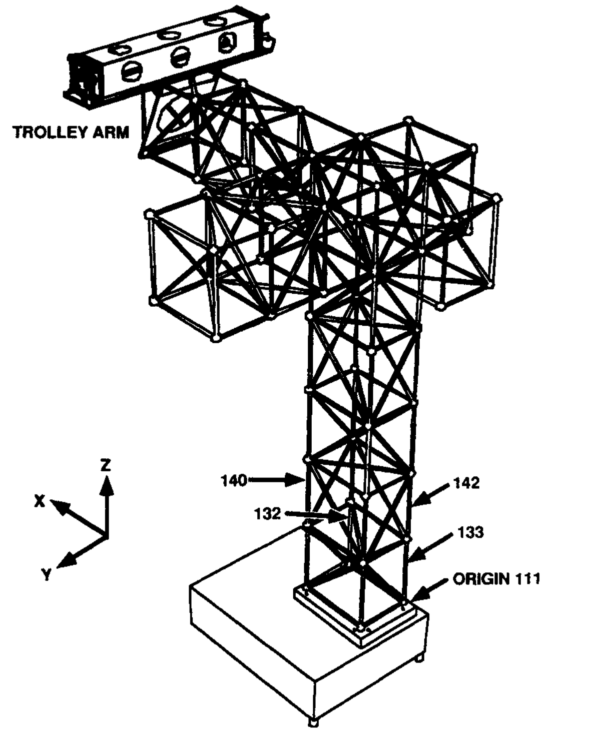

Fig. 1 Jet Propulsion Laboratory's control structure interaction (CSI) Phase B test bed.

Examples and Discussions

The discussion in this section will center around a quantitative analysis of several of the relations established in the previous section for approximating modal frequencies and dampings of the physical system depicted in Fig. 1. A slight digression into an analysis of alternative techniques for eigenvalue approximation motivated by the analytical expression derived in Theorem 1 will be made. The extension of the Ritz reduction method to the multiple damper case will also be briefly considered.

The physical system consists of a vertical tower of length 2.5 m with two horizontal arms at right angles to each other attached at the top of the tower. The tower and arms are trusses built from a total of 14 rectangular bays, each with the dimensions $40.6 \times 40.6 \times 28.7$ cm. Attached to one of the horizontal arms is the optical motion compensation system, which is encased in a flexure mounted frame. (For more details, see Ref. 3) The system was modeled using NASTRAN with beam elements for the individual struts. The resulting 666-DOF system was reduced by Guyan reduction to 249 DOF. This reduced model constitutes the nominal full-order model for the numerical investigations in this study. No damping is assumed in the nominal model.

In the first study, damping is introduced by a single viscous element located at the strut location 133, as indicated in Fig. 1. The stiffness and damping coefficients of this damper are shown in Tables 1 and 2, which are labeled as k_{p133} and k_{v133}, respectively. Note that the damper stiffness used is only one-tenth of the stiffness of the regular strut that it replaces.

In Table 1, the first column contains the first 12 modal frequencies of the undamped nominal model, which are labeled as $\omega_1, \ldots, \omega_{12}$. The next three columns contain the corresponding frequency estimates of the damped system obtained through solving the eigenvalue problem based on the full-order (249 modes), Ritz reduced, and modally reduced models, respectively. The Ritz reduced model was obtained by retaining eigenvectors corresponding to the first 12 undamped modes augmented with the Ritz vector corresponding to the damping element at the specified location. The modally reduced model was obtained by retaining the first 13 undamped modes. The corresponding damping estimates based on these three models are shown in Table 2. It is shown that the Ritz reduced models lead to superior estimates of both frequency and damping when compared with the modally reduced model. The modally reduced model does rather poorly in its prediction of damping in several modes; in some instances, it provides estimates that are an order of magnitude smaller than the "true" damping values as determined by the full damped model. The Ritz reduced model, which contains the same number of modes, consistently captures four or more digits of accuracy in both frequency and damping.

A computation based on the relations in Theorem 4 shows that if the damping is removed from this model, i.e., setting $k_{v133} = 0$, then independent of the value of the stiffness, the Ritz reduced model will produce more accurate estimates of all frequencies than the modally reduced model. Corollary 5 then implies that this result extends to small damping values. Hence, the qualitative comparison results between the modally reduced and Ritz reduced models in the table are predicted from Theorem 4 and Corollary 5. Interestingly, these same conclusions have been established for any position of the damper and for any modally truncated model obtained by augmenting the first 12 modes with a single arbitrary higher frequency mode. This result was established by exhaustively checking the conditions in Theorem 4 over all cases. Unfortunately we have not been able to derive this result analytically.

Table 3 compares the actual eigenvalue error resulting from the Ritz reduced model to the a posteriori estimates obtained from the relation (20):

$$|\lambda^* - \lambda_{\mathrm{Ritz}}| < \rho$$

with

$$\rho = 2f(\lambda_{\mathrm{Ritz}})/f'(\lambda_{\mathrm{Ritz}})$$

Here, λ^* is the true eigenvalue, and λ_{Ritz} denotes the eigenvalue estimate obtained from the Ritz reduction. Using a conservative bound on $f''(\lambda)$ in a neighborhood of ρ about λ_{Ritz}, the sufficient condition (19) for the previous estimate to be valid was verified for each of the 12 eigenvalues. As can be seen from the exact and a posteriori errors in the table, these bounds are fairly tight. These bounds can be tightened by using the computed value of $(1-\sqrt{1-2h})/h$ [cf. Eq. (19)] instead of the upper bound of 2 as was used in the table.

Figure 2 explores a comparison of the relations (17) and (18). Based on these a priori estimates, it is conjectured that an error improvement in the kth mode estimate by a factor of $\mathcal{O}(\omega_k^3/\omega_{r+1}^4)$ results from augmenting the modally truncated model with a Ritz vector. For this study, the damper was

Table 3 Actual and a posteriori estimate errors of the predicted damped eigenvalues based on the Ritz reduced model

Mode number	Actual error	A posteriori error estimate
2	2.3566817E-06	4.7135437E-06
3	7.3560831E-06	1.4712053E-05
4	4.0155174E-05	8.0310333E-05
5	8.0278777E-05	1.6055778E-04
6	4.7907805E-04	9.5816217E-04
7	4.9352941E-05	9.8705926E-05
8	2.3247126E-03	4.6494426E-03
9	1.3373237E-02	2.6746154E-02
10	1.9525004E-02	3.9048746E-02
11	2.2659903E-03	4.5319626E-03
12	1.1885087E-03	2.3770109E-03

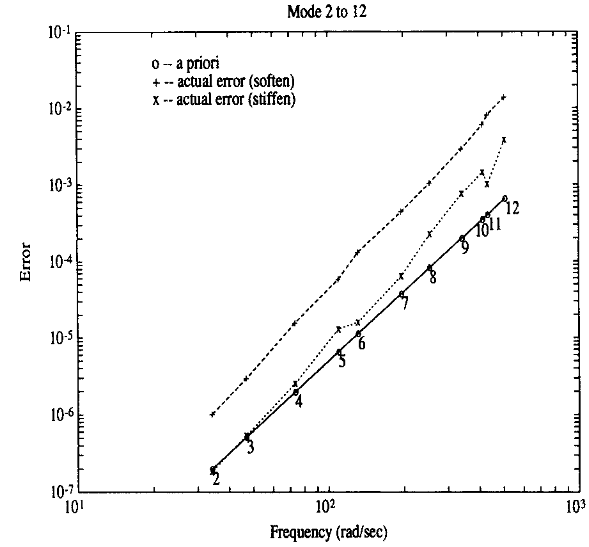

Fig. 2 Comparison of actual errors vs a priori error estimates.

placed at location 133 with the damping coefficient fixed at 400 lb/in. Two different values of damper stiffness were considered for investigating this conjecture. In the first case, the damper is softened to one-tenth the stiffness of the regular strut as in the first study. In the second case, it is stiffened to 10 times of the stiffness of the regular strut.

The empirically derived ratios for these two cases are shown against the curve $\omega_k^3/\omega_{r+1}^4$ in a log-log plot as k is incremented through the first 12 frequencies, $k = 1, \ldots, 12$. The a priori predictions are better for the stiffened case than for the softened case. However, the trends in the relative error growth in both cases coincide fairly well with the predicted a priori trends. It should be noted that the estimate $\mathcal{O}(\omega_k^3/\omega_{r+1}^4)$ is a conjecture based on taking the ratio of two a priori estimates and, thus, is not a true a priori estimate of the ratio.

In optimization applications wherein the objective is to maximally damp selected structural modes, it is extremely important, especially for large systems with many degrees of freedom, to be able to efficiently solve large eigenvalue problems as the damper characteristics are varied. These problems motivated the analysis of the Ritz reduction method of the present paper. This analysis also has application to alternative methods for solving these eigenvalue problems. As an example of this, eigenvalue derivatives (also known as Jacobi's method) have been proposed[15] as a technique for obtaining damping estimates in the case where damping elements produce "small" perturbations in the nominal structure.

This method has a transparent interpretation with respect to the function $d(\lambda)$ derived in Theorem 1; as we shall see, simple improvements to the method can be made. To show this connection, recall the function $f(\lambda) = (\lambda^2 + \omega_k^2)d(\lambda)$ introduced earlier. This function may be written as,

$$f(\lambda) = \lambda^2 + \omega_k^2 + (\lambda k_v + k_p)b_k^2 + r_k(\lambda)$$

where

$$r_k(\lambda) = (\lambda^2 + \omega_k^2)(\lambda k_v + k_p) \sum_{j \neq k} \frac{b_j^2}{\lambda^2 + \omega_j^2}$$

Theorem 1 implies (so long as $b_k \neq 0$) that the zeros of f and d coincide. Thus, the perturbed ith eigenvalue resulting from the damper is the zero of f in a neighborhood of $i\omega_k$. As a first approximation to finding this perturbation, set $r_k = 0$ in the preceding equations. A simple quadratic expression results from this approximation, and the approximate frequency $\hat{\omega}_k$ and damping $\hat{\xi}_k$ for the ith mode are obtained as

$$\hat{\omega}_k = (\omega_k^2 + k_p b_k^2)^{1/2}, \qquad \hat{\xi}_k = \frac{k_v b_k^2}{2\hat{\omega}_k}$$

Table 4 Frequency prediction (Hz) of the damped system ($k_{p133} = 5000.72$ lb/in., $k_{v133} = 400$ lb-s/in.)

Mode number	Newton method	Quadratic approximation	Jacobi method
1	0.7417	0.7417	0.7425
2	5.2315	5.2288	5.2986
3	7.1353	7.1392	7.2364
4	11.5462	11.5467	11.4196
5	17.4846	17.4848	17.2947
6	20.9812	20.9799	20.5034
7	31.1477	31.1477	31.1301
8	40.6585	40.6585	40.3300
9	54.6942	54.6942	54.1276
10	65.9101	65.9101	65.5647
11	68.8930	68.8930	68.8636
12	80.8419	80.8419	80.8343

Table 5 Damping prediction (Hz) of the damped system ($k_{p133} = 5000.72$ lb/in., $k_{v133} = 400$ lb-s/in.)

Mode number	Newton method	Quadratic approximation	Jacobi method
1	0.0234	0.0234	0.0014
2	8.2316	8.5383	0.7173
3	6.8431	6.7901	1.2397
4	4.9212	4.9171	1.4483
5	1.4621	1.4604	0.7247
6	3.3098	3.3156	1.8957
7	0.0440	0.0440	0.0473
8	0.4373	0.4373	0.7755
9	0.4033	0.4033	1.2416
10	0.1710	0.1710	0.7308
11	0.0146	0.0146	0.0602
12	0.0023	0.0023	0.0161

This approximation is identical to the Jacobi (or eigenvalue derivative) approximation obtained in Ref. 15. It is apparent that the accuracy of this approximation is dependent on the contribution of the term r_k in a neighborhood of $i\omega_k$.

Since $r(i\omega_k) = 0$, the Jacobi approximation can be viewed as a zeroth-order approximation to r_k in a neighborhood of $i\omega_k$. A better approximation to f is obtained by using a quadratic approximation of r_k. The resulting approximation to the rational function f is again quadratic and is thus solvable in closed form. This second-order expansion is easily obtained by computing $r'(i\omega_k)$ and $r''(i\omega_k)$. Further fidelity of eigenvalue approximations can be developed by the Newton iteration scheme

$$\lambda^{(n+1)} = \lambda^{(n)} - f(\lambda^{(n)})/f'(\lambda^{(n)})$$

where $\lambda^{(n)}$ denotes the nth iterate.

These three schemes for eigenvalue approximation are compared in Tables 4 and 5. The data shown in these tables are generated with the same locations and damper parameters used in the first study (cf. Tables 1 and 2). The Jacobi method is observed to produce inaccurate damping estimates, similar to the inaccuracies resulting from the modally reduced model. The simple quadratic approximation scheme performs extremely well in comparison. It is noted that this scheme is quite different from the second-order eigenvalue derivative approximation in Ref. 15. (In fact, only very marginal improvement in eigenvalue estimates are gained beyond the Jacobi approximation with the inclusion of the second derivative information.) The Newton method exhibited excellent convergence properties, with a maximum of five iterations per eigenvalue to achieve 10 digits of accuracy. The Newton algorithm was initialized with the nominal system eigenvalues, and typically as few as three iterations were required for convergence. (Details of this method are described in Ref. 19.)

The Ritz reduction method was also tested with a multiple damper configuration consisting of three dampers. For simplicity, the stiffness and damping coefficients of all three viscous dampers, denoted as k_p and k_v, are assumed to be the same as shown in Tables 6 and 7. The three damper locations are listed in Tables 6 and 7 and shown in Fig. 1. The model for the multiple damper case is a straightforward generalization of the single damper setting. The second-order form for this model in modal coordinates [cf. Eq. (10)] is

Table 6 Frequency prediction (Hz) of the damped system with three passive dampers at locations 132, 140, and 142 ($k_p = 8000$ lb/in., $k_v = 320$ lb-s/in.)

Mode number	249 modes (full order)	12 modes plus 3 Ritz vectors	15 modes (truncation)
1	0.7420	0.7420	0.7425
2	5.2940	5.2940	5.3262
3	7.0376	7.0376	6.9540
4	10.4862	10.4862	10.4493
5	17.4386	17.4386	17.3444
6	20.8236	20.8236	20.7055
7	31.2231	31.2231	31.0481
8	40.6380	40.6380	40.3183
9	55.1606	55.1649	54.1306
10	65.7059	65.7060	65.6915
11	68.9211	68.9220	68.8670
12	80.8402	80.8403	80.8359

Table 7 Damping prediction (%) of the damped system with three passive dampers at locations 132, 140, and 142 ($k_p = 8000$ lb/in., $k_v = 320$ lb-s/in.)

Mode number	249 modes (full order)	12 modes plus 3 Ritz vectors	15 modes (truncation)
1	0.0179	0.0179	0.0012
2	4.5744	4.5744	0.6125
3	25.5358	25.5357	2.3228
4	32.6380	32.6379	5.5664
5	0.9033	0.9034	0.4066
6	1.3197	1.3197	0.5709
7	0.5013	0.5016	0.5031
8	0.5415	0.5421	0.6403
9	0.8620	0.8772	3.4119
10	0.0087	0.0089	0.0319
11	0.0339	0.0355	0.1691
12	0.0021	0.0021	0.0084

$$\ddot{q} + BK_vB^T\dot{q} + (D + BK_pB^T)q = \Phi^T f$$

where K_p and K_v are diagonal matrices containing the stiffness and damping parameters for each of the dampers, and the matrix B is the multiple damper analog of the vector b in the single damper case. Accordingly, the corresponding augmentation of Ritz vectors involves appending the column vectors of the matrix $D^{-1}B$ to the retained modes. The results of this Ritz reduction technique are shown in Tables 6 and 7, and as in the single damper case, dramatic superiority in approximated eigenvalues is achieved over the modally truncated model approximation.

In a future paper, the use of these reduction techniques in optimization loops for optimizing damper location and parameter values with respect to various performance metrics will be addressed. This work will also encompass an extension of the single damper Newton iteration scheme to the multiple damper setting for selected eigenvalue and eigenvector updates.

Appendix: Proofs of Theorems 1, 2, 3, 4, and 6

Proof of Theorem 1

First observe that we can write for $\lambda \neq \pm i\omega_j$

$$M(\lambda) = (\lambda^2 + D)[I + (\lambda k_v + k_p)(\lambda^2 + D)^{-1}bb^T]$$

Upon using the identity

$$\det(I_{n\times n}+XY)=\det(I_{m\times m}+YX)$$

where X and Y are $n\times m$ and $m\times n$ matrices, respectively, and $I_{k\times k}$ denotes the $k\times k$ identity matrix, we have

$$\det M(\lambda)=\det(\lambda^2+D)[1+(\lambda k_v+k_p)\langle b,(\lambda^2+D)^{-1}b\rangle]$$

$$=\left(\prod_{j\neq k}(\lambda^2+\omega_j^2)\right)\Bigg[(\lambda^2+\omega_k^2)+(\lambda k_v+k_p)\left\{b_k^2+(\lambda^2+\omega_k^2)\sum_{j\neq k}\frac{b_j^2}{\lambda^2+\omega_j^2}\right\}\Bigg]$$

Since det $M(\cdot)$ is continuous (in fact, analytic),

$$\det M(\pm i\omega_k)=\left(\prod_{j\neq k}(\omega_j-\omega_k)\right)(\pm i\omega_k k_v+k_p)b_k^2$$

Because $\omega_j\neq\omega_k$ for $j\neq k$, and k_p and k_v are both real, with at least one of them nonzero, it follows that $\pm i\omega_k$ solves Eq. (15) if and only if $b_k=0$. By assumption, $b_k\neq 0$, and the first assertion of the theorem is proved.

Noting that

$$\det M(\lambda)=\prod_{j=1}^{n}(\lambda^2+\omega_j^2)d(\lambda)$$

with $\det M(\pm i\omega_k)\neq 0$ for all k, it follows that the zeros of $\det M(\cdot)$ are precisely the zeros of $d(\cdot)$.

Proof of Theorem 2

The first assertion of the theorem follows from applying Theorem 1 to the eigenvalue problem

$$\det(\lambda^2 I+U^T[D+(\lambda k_v+k_p)bb^T]U)=0$$

To get the form of the error in Eq. (16), we first compute

$$\langle Df_{r+1}f_{r+1}\rangle=\frac{1}{\sum_{j=r+1}^{n}b_j^2/\omega_j^4}\left\langle\sum_{j=r+1}^{n}b_je_j,\ \sum_{i=r+1}^{n}(b_i/\omega_i^2)e_i\right\rangle$$

$$=\frac{\sum_{j=r+1}^{n}b_j^2/\omega_j^2}{\sum_{j=r+1}^{n}b_j^2/\omega_j^4}$$

$$\langle b,f_{r+1}\rangle^2=\frac{1}{\sum_{j=r+1}^{n}b_j^2/\omega_j^4}\left(\sum_{j=r+1}^{n}b_j^2/\omega_j^2\right)^2$$

With these relations note that

$$d(\lambda)-d_u(\lambda)=(\lambda k_v+k_p)\left\{\sum_{j=r+1}^{n}\frac{b_j^2}{\omega_j^2+\lambda^2}-\frac{<b,f_{r+1}>^2}{<Df_{r+1},f_{r+1}>+\lambda^2}\right\}$$

$$=(\lambda k_v+k_p)\sum_{j=r+1}^{n}b_j^2\left\{\sum_{l=r+1}^{n}\frac{\gamma_l}{\lambda^2+\omega_l^2}-\frac{\mu_1}{1+\lambda^2\mu_2/\mu_1}\right\}$$

Now since $\omega_j^4\geq\omega_j^2\omega_{r+1}^2$ for $j\geq r+1$,

$$\frac{\mu_1}{\mu_2}=\frac{\sum_{j=r+1}^{n}\gamma_j/\omega_j^2}{\sum_{j=r+1}^{n}\gamma_j/\omega_j^4}>\omega_{r+1}^2$$

Hence, $d-d_u$ is analytic in $\{\lambda:|\lambda|^2<\omega_{r+1}^2\}$. Therefore expanding $d-d_u$ in a Taylor series about zero we obtain

$$d(\lambda)-d_u(\lambda)=(\lambda k_v+k_p)\left\{\mu_1\sum_{k=0}^{\infty}(\lambda^2\mu_2/\mu_1)^k-\sum_{j=r+1}^{n}\gamma_j/\omega_j^2\sum_{k=0}^{\infty}(\lambda^2/\omega_j^2)^k\right\}\tag{A1}$$

For $k=0$, the coefficient in the expansion of the term in the braces is

$$\mu_1-\sum_{j=r+1}^{n}\gamma_j/\omega_j^2=0$$

And similarly, the coefficient for $k=1$ is

$$\mu_2-\sum_{j=r+1}^{n}\gamma_j/\omega_j^4=0$$

Using the fact that these terms are zero, Eq. (16) is obtained after factoring out the term λ^4 in the infinite series in Eq. (A1) and rewriting the resulting expression as a rational function.

Proof of Theorem 3

Introduce the functions f and f_u by $f=(\lambda^2+\omega_k^2)d$ and $f_u=(\lambda^2+\omega_k^2)d_u$. Theorem 1 implies that f and d have the same set of zeros and also that f_u and d_u have the same set of zeros. For $|k_v|$ and $|k_p|$ sufficiently small, the continuity of $\lambda_u(\cdot)$ implies that f_u has a zero in a neighborhood of radius δ about $i\omega_k$ with $\delta<\min\{\omega_k-\omega_{k-1},\omega_{k+1}-\omega_k\}$ and $\delta<<\omega_k$. Now since $f_u(\lambda_u)=0$, we have

$$f(\lambda_u)=f(\lambda_u)-f_u(\lambda_u)+f_u(\lambda_u)$$
$$=(\lambda_u^2+\omega_k^2)[d(\lambda_u)-d_u(\lambda_u)]\tag{A2}$$

Theorem 2 implies

$$|f(\lambda_u)|\leq|\lambda_u|^4|\lambda_u^2+\omega_k^2|(|\lambda_u|k_v+|k_p|)\sum_{j=r+1}^{n}b_j^2\left\{\frac{\mu_2^2/\mu_1}{|1+\lambda_u^2\mu_2/\mu_1|}+\sum_{l=r+1}^{n}\frac{\gamma_l/\omega_l^6}{|1+\lambda_u^2/\omega_l^2|}\right\}$$

To obtain a simpler bound on $f(\lambda_u)$ note that if $|\lambda_u-i\omega_k|<\delta$, then

$$|1+\lambda_u^2\mu_2/\mu_1|>1-[\omega_k^2+2\delta\omega_k+\delta^2]\mu_2/\mu_1\tag{A3}$$

$$>1-\frac{\omega_k^2+2\delta\omega_k+\delta^2}{\omega_{r+1}^2}\tag{A4}$$

and similarly for $l\geq r+1$

$$|1+\lambda_u^2/\omega_l^2|>1-\frac{\omega_k^2+2\delta\omega_k+\delta^2}{\omega_{r+1}^2}$$

Denote the constant on the right by β, and note that for $\delta<<\omega_k$, $\beta\approx(\omega_{r+1}^2-\omega_k^2)/\omega_{r+1}^2$. From this we obtain the estimates

$$\sum_{j=r+1}^{n}\frac{\gamma_j/\omega_j^6}{|1+\lambda_u^2/\omega_j^2|}\leq\frac{1}{\beta\omega_{r+1}^6}\sum_{j=r+1}^{n}\gamma_j\tag{A5}$$

$$\leq\frac{1}{\beta\omega_{r+1}^6}\tag{A6}$$

$$\frac{\mu_2^2/\mu_1}{|1+\lambda_u^2\mu_2/\mu_1|}\leq\frac{\mu_2}{\beta\omega_{r+1}^2}\tag{A7}$$

$$\leq\frac{1}{\beta\omega_{r+1}^6}\tag{A8}$$

Thus,

$$|f(\lambda_u) \le |\lambda_u^2 + \omega_k^2| \frac{|\omega_k + \delta|^4}{\beta\omega_{r+1}^6} |(\omega_k + \delta)k_v + k_p| \sum_{j=r+1}^{n} b_j^2$$

The Newton-Kantorovich theorem implies that a sufficient condition for a solution to $f(\lambda) = 0$ in a neighborhood $N(\lambda_u, \rho)$ of radius ρ about λ_u is for the inequality

$$\frac{|f(\lambda_u)|}{|f'(\lambda_u)|^2} |f''(\lambda)| < \frac{1}{2} \tag{A9}$$

to hold for all λ in $N(\lambda_u, \rho)$ with

$$\rho > \frac{1}{h}(1 - \sqrt{1 - 2h}) \frac{|f(\lambda_u)|}{|f'(\lambda_u)|}$$

where h is the supremum over $\lambda \in N(\lambda_u, \rho)$ of the left side of Eq. (A9). Since $h \to (1 - \sqrt{1 - 2h})/h$ is an increasing function on (0, 1/2), it suffices for Eq. (A9) to hold on $N(\lambda_u, \rho_0)$, where $\rho_0 = 2|f(\lambda_u)|/|f'(\lambda_u)|$. Taking first and second derivatives of f at λ_u with the assumption of "small" $|k_v|$ and $|k_p|$, the approximations $|f'(\lambda_u)| \approx 2|\lambda_u|$ and $f''(\lambda_u) \approx 2$ are valid. Thus, the hypotheses of the Newton-Kantorovich theorem are satisfied and

$$|\lambda_u - \lambda| < \frac{2|f(\lambda_u)|}{|f'(\lambda_u)|}$$

Using the bound (A9) with the estimates $|f'(\lambda_u)| \approx 2|\lambda_u|$ and $\delta << \omega_k$ leads to Eq. (17), completing the proof of the theorem.

Proof of Theorem 4

First assume that $b_i \neq 0$ for all i, and that $k_p > 0$. We note the following two relationships for each index j:

$$\omega_j^2 < t^j(k_p),\ t_u^j(k_p),\ t_s^j(k_p) < \omega_{j+1}^2 \tag{A10}$$

$$t^j(k_p) < t_u^j(k_p),\ t_s^j(k_p) < \omega_{j+1}^2 \tag{A11}$$

Relation (A10) is a consequence of $k_p > 0$ and a result in Ref. 16 (p. 462). Relation (A11) is a consequence of the minimax theorem[16] for eigenvalues of symmetric matrices.

Algebraic manipulations as in the proof of Theorem 2 lead to

$$g_u(t) - g_s(t) = k_p \sum_{j=r+1}^{n} b_j^2 \left\{ \frac{\mu_1}{1 - t\mu_2/\mu_1} - \frac{\gamma_s}{d_s - t} \right\}$$

Recalling that $t < \omega_{r+1}^2$ implies $1 - t\mu_2/\mu_1 > 0$, it follows that the set

$$T = \{t : 0 < t < \omega_{r+1}^2 \quad \text{and} \quad g_u(t) - g_s(t) > 0\}$$

is obtained by solving the inequality

$$\mu_1(\omega_s^2 - t) - \gamma_s(1 - t\mu_2/\mu_1) > 0 \tag{A12}$$

for $t \ge 0$. Now

$$\gamma_s - \mu_1\omega_s^2 = \gamma_s - \sum_{j=r+1}^{n} \frac{\gamma_j \omega_s^2}{\omega_j^2} < 0$$

and from this latter inequality we may deduce that a neighborhood of zero is contained in T. Furthermore, if

$$\gamma_s\mu_2/\mu_1 - \mu_1 > 0$$

then $T \supset [0, \omega_{r+1}^2)$. And if $\gamma_s\mu_2/\mu_1 - \mu_1 < 0$, we still have $T \supset [0, t^*)$ with

$$t^* = \frac{\mu_1\omega_s^2 - \gamma_s}{\mu_1 - \gamma_s\mu_2/\mu_1}$$

Next, suppose $t^* \in [\omega_k^2, \omega_{k+1}^2]$. Then for any index $j < k$, it follows that $g_u > g_s$ on $(\omega_j^2, \omega_{j+1}^2)$. Also recall that since $k_p > 0$, g, g_u, and g_s are all increasing on $(\omega_j^2, \omega_{j+1}^2)$. Noting Eq. (A10), observe that $g_u < g$ with $g_u < 0$ on $[t^j(k_p), t_u^j(k_p)]$. However, since $g_s < g_u$ on $(\omega_j^2, \omega_{j+1}^2)$, it follows from the monotonicity of g_s that $t_u^j(k_p) < t_s^j(k_p)$. Hence, Eqs. (24) and (25) hold for the case $k_p > 0$. The same argument shows that this inequality is reversed for indices $j > k$.

The case $k_p < 0$ is proved in the same manner using the fact that g, g_u and g_s are all monotonically decreasing functions between their poles.

To remove the restriction that $b_i \neq 0$ for all i, we note that the zeros of g, g_u, and g_s are all continuous functions of the vector b (since they are the eigenvalues of their corresponding models, cf. Theorem 1) Thus, the relationships (24–26) hold in the limit for any Cauchy sequence of vectors $\{b^n\}$; in particular, independently of the values of the components b_i.

Proof of Theorem 6

The corresponding function g_{b^*} for the model (27) [compare Eq. (22)] is obtained as

$$g_{b^*}(t) = k_p \sum_{j=1}^{n} b_j^2 \left\{ \sum_{j=1}^{r} \frac{\gamma_j}{\omega_j^2 - t} - \frac{1}{\sum_{j=r+1}^{n} \gamma_j(\omega_j^2 - t)} \right\}$$

In the case that $k_p > 0$, arguing in the same manner as in Theorem 4, the set

$$T = \{t : 0 < t < \omega_{r+1}^2 \quad \text{and} \quad g_u(t) - g_{b^*}(t) > 0\}$$

is obtained by solving the inequality

$$\frac{\mu_1}{1 - t\mu_2/\mu_1} - \frac{1}{\sum_{j=r+1}^{n} \gamma_j(\omega_j^2 - t)} > 0 \tag{A13}$$

Because the denominators in both terms are positive for $t < \omega_{r+1}^2$, Eq. (A13) holds if and only if

$$t(\mu_2/\mu_1 - \mu_1) > 1 - \mu_1 \sum_{j=r+1}^{n} \gamma_j\omega_j^2$$

We will show that this inequality holds for $t > 0$. To prove this, it will first be shown that

$$\mu_2/\mu_1 - \mu_1 \ge 0$$

Now since $\mu_1 > 0$, to establish the previous inequality it is sufficient to show that $\mu_2 - \mu_1^2 \ge 0$. Note that

$$\mu_1^2 = \left\{ \sum_{j=r+1}^{n} \frac{\gamma_j}{\omega_j^2} \right\}^2 \tag{A14}$$

$$= \left\{ \sum_{j=r+1}^{n} \frac{\sqrt{\gamma_j}}{\omega_j^2} \sqrt{\gamma_j} \right\}^2 \tag{A15}$$

Therefore by the Schwarz inequality

$$\mu_1^2 \le \left(\sum_{j=r+1}^{n} \gamma_j/\omega_j^2 \right) \left(\sum_{j=r+1}^{n} \gamma_j \right) \tag{A16}$$

$$= \mu_2 \tag{A17}$$

since $\Sigma\gamma_j = 1$. Thus, $\mu_2/\mu_1 - \mu_1 \ge 0$. Next we claim that

$$1 - \mu_1 \sum_{j=r+1}^{n} \gamma_j\omega_j^2 < 0 \tag{A18}$$

or, equivalently,

$$\frac{1}{\sum_{j=r+1}^{n} \gamma_j/\omega_j^2} < \sum_{j=r+1}^{n} \gamma_j \omega_j^2 \tag{A19}$$

This inequality follows from the convexity (Rudin, W., *Real and Complex Variables*, McGraw-Hill, New York, 1974) of the function $h(x) = 1/x$ on the positive real axis: Since $h(x)$ is convex,

$$h\left(\sum_{j=r+1}^{n} \gamma_j \omega_j^2\right) < \sum_{j=r+1}^{n} \gamma_j h(\omega_j^2)$$

for any nonnegative numbers γ_j and ω_j^2 with $\Sigma\, \gamma_j = 1$. Taking reciprocals in this inequality, we obtain Eq. (A19).

With these two claims established, it follows that Eq. (A13) holds for

$$t > \frac{1 - \mu_1 \sum_{j=r+1}^{n} \gamma_j \omega_j^2}{(\mu_2/\mu_1 - \mu_1)}$$

The right side of this inequality is negative. Hence, for $k_p > 0$ and $t > 0$, it follows that $g_u > g_{b*}$. For $k_p < 0$, the reverse inequality holds, i.e., $g_{b*} > g_u$. Arguing precisely as in Theorem 4 proves the theorem.

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